REVIEW QUESTIONS - CHAPTER 5

5.1. True or false: A small residual $|| f(x) ||$ guarantees an accurate solution of a system of nonlinear equations $f(x)=0$.

5.2. True or false: Newton's method is an example of a fixed-point iteration scheme.

5.3. True or false: If an iterative method for solving a nonlinear equation gains more than one bit of accuracy per iteration, then it is said to have a superlinear convergence rate.

5.4. True or false: For a given fixed level of accuracy, a superlinearly convergent iterative method always requires fewer iterations than a linearly convergent method to find a solution to that level of accuracy.

5.5. Suppose you are using an iterative method to solve a nonlinear equation $f(x) = 0$ for a root that is ill-conditioned, and you need to choose a convergence test. Would it be better to terminate the iteration when you find an iterate x_k for which $|| f(x_k) ||$ is small, or when $|| x_k - x_{k-1} ||$ is small? Why?

5.6. (a) What is meant by a bracket for a nonlinear function in one dimension?(b) What does this concept have to do with zero finding?

5.7. For root finding problems, why must we use an absolute rather than a relative condition number in assessing sensitivity?

5.8. (a) What is the definition of the convergence rate r of an iterative method?

(b) Is it possible to have a cubically convergent method (r = 3) for finding a zero of a function?

(c) If not, why, and if so, how might such a scheme be derived?

5.9. If the errors at successive iterations of an iterative method are as follows, how would you characterize the convergence rate?

(a) $10^{-2}, 10^{-4}, 10^{-8}, 10^{-16}, \ldots$

(b) $10^{-2}, 10^{-4}, 10^{-6}, 10^{-8}, \ldots$

5.10. What condition ensures that the bisection method will find a zero of a continuous nonlinear function f in he interval $[a, b]$?

5.11. (a) If the bisection method for finding a zero of a function $f: \mathbf{R} \rightarrow \mathbf{R}$ starts with an initial bracket of length 1, what is the length of the interval containing the root after six iterations?

(c) Do you need to know the particular function f to answer the question in part *a*?

(d) If we assume that it is started with a bracket for the solution in which there is a sign change, is the convergence rate of the bisection method dependent on whether the solution sought is a simple root or a multiple root? Why?

5.12. Suppose you are using the bisection method to find a zero of a nonlinear function, starting with an initial bracketing interval $[a, b]$. Give a general expression for the number of iterations that will be required to achieve an error tolerance of *tol* for the length of the final bracketing interval.

5.13. What is meant by a *quadratic* convergence rate for an iterative method?

5.14. If an iterative method squares the error every *two* iterations, what is its convergence rate *r*?

5.15. (a) What does it mean for a root of an equation to be a *multiple* root?

(b) What is the effect of a multiple root on the convergence rate of the bisection method? (c) What is the effect of a multiple root on the convergence rate of Newton's method?

5.16. Which of the following behaviors are possible in using Newton's method for solving a nonlinear equation?

(a) It may converge linearly.

(b) It may converge quadratically.

(c) It may not converge at all.

5.17. What is the convergence rate for Newton's method for finding the root x = 2 of each of the following equations?

(a) $f(x) = (x - 1)(x - 2)^2 = 0$

(b) $f(x) = (x - 1)^2(x - 2) = 0$

5.18. *(a)* What is meant by a *fixed point* of a function $g(x)$?

(b) Given a nonlinear equation $f(x) = 0$, how can you determine an equivalent fixed-point problem, that is, a function $g(x)$ such that a fixed point x of g is a solution to the nonlinear equation $f(x) = 0$?

(c) Specifically, what function $g(x)$ results from this approach?

5.19. In using the secant method for solving a one-dimensional nonlinear equation,

(a) How many starting guesses for the solution are required?

(b) How many new function evaluations are required per iteration?

5.20. Let g: $\mathbf{R} \rightarrow \mathbf{R}$ be a smooth function having a fixed point x^*.

(a) What condition determines whether the iteration scheme $x_{k+1} = g(x_k)$ is locally convergent to x^*?

(b) What is the convergence rate?

(c) What additional condition implies that the convergence rate is quadratic?

(d) Is Newton's method for finding a zero of a smooth function f: $\mathbf{R} \rightarrow \mathbf{R}$ an example of such a fixed-point iteration scheme? If so, what is the function g in this case? If not, then explain why not.

5.21. In bracketing a zero of a nonlinear function, one needs to determine if two function values, say $f(a)$ and $f(b)$, differ in sign. Is the following a good way to test for this condition: if *(f(a) * f(b) < 0) . . .?* Why?

5.22. Let g: $\mathbf{R} \rightarrow \mathbf{R}$ be a smooth function, and let x^* be a point such that $g(x^*) = x^*$.

(a) State a general condition under which the iteration scheme $x_{k+1} = g(x_k)$ converges quadratically to x^*, assuming that the starting guess x_0 is close enough to x^*.

(b) Use this condition to prove that Newton's method is locally quadratically convergent to a simple zero x^* of a smooth function f: $\mathbf{R} \rightarrow \mathbf{R}$.

5.23. List one advantage and one disadvantage of the secant method compared with the bisection method for finding a simple zero of a single nonlinear equation.

5.24. List one advantage and one disadvantage of the secant method compared with Newton's method for solving a nonlinear equation in one dimension.

5.25. The secant method for solving a onedimensional nonlinear equation uses linear interpolation of the given function at two points. Interpolation at more points by a higher-degree polynomial would increase the convergence rate of the iteration.

(a) Give three reasons why such an approach might not work well.

(b) What alternative approach using higher-degree interpolation in this context avoids these difficulties?

5.26. For solving a one-dimensional nonlinear equation, how many function or derivative evaluations are required per iteration of each of the following methods?

(a) Newton's method

(b) Secant method

5.27. Rank the following methods 1 through 3, from slowest convergence rate to fastest convergence rate, for finding a simple root of a nonlinear equation in one dimension:

(a) Bisection method
(b) Newton's method
(c) Secant method

5.28. In solving a nonlinear equation in one dimension, how many bits of accuracy are gained per iteration of

(a) Bisection method?
(b) Newton's method?

5.29. In solving a nonlinear equation $f(x) = 0$, if you assume that the cost of evaluating the derivative $f'(x)$ is about the same as the cost of evaluating $f(x)$, how does the cost of Newton's method compare with the cost of the secant method per iteration?

5.30. What is meant by inverse interpolation? Why is it useful for root finding problems in one dimension?

5.31. Suppose that you are using fixed-point iteration based on the fixed-point problem $x=g(x)$ to find a solution x^* to a nonlinear equation $f(x) = 0$. Which would be more favorable for the convergence rate: a horizontal tangent of g at x^* or a horizontal tangent of f at x^*? Why?

5.32. Suggest a procedure for safeguarding the secant method for solving a one-dimensional nonlinear equation so that it will still converge even if started far from a root.

5.33. For what type of function is linear fractional interpolation a particularly good choice of zero finder?

5.34. Each of the following methods for computing a root of a nonlinear equation has the same asymptotic convergence rate. For each method, specify a situation in which that method is particularly appropriate.

(a) Regular quadratic interpolation
(b) Inverse quadratic interpolation
(c) Linear fractional interpolation

5.35. State at least one method for finding all the zeros of a polynomial, and discuss its advantages and disadvantages.

5.36. Does the bisection method generalize to finding zeros of multidimensional functions? Why?

5.37. For solving an n-dimensional nonlinear equation, how many scalar function evaluations are required per iteration of Newton's method?

5.38. Relative to Newton's method, which of the following factors motivate secant updating methods for solving systems of nonlinear equations?

(a) Lower cost per iteration
(b) Faster convergence rate
(c) Greater robustness far from solution
(d) Avoidance of computing derivatives

5.39. Give two reasons why secant updating methods for solving systems of nonlinear equations are often more efficient than Newton's method despite converging more slowly.