REVIEW QUESTIONS - CHAPTER 3

3.1. True or false: A linear least squares problem always has a solution.
3.2. True or false: Fitting a straight line to a set of data points is a linear least squares problem, whereas fitting a quadratic polynomial to the data is a nonlinear least squares problem.
3.3. True or false: At the solution to a linear least squares problem Ax ≈ b, the residual vector r = *b*-Ax is orthogonal to span(A).
3.4. True or false: An overdetermined linear least squares problem Ax ≈ b always has a unique solution *x* that minimizes the Euclidean norm of the residual vector r = *b-Ax.*
3.5. True or false: In solving a linear least squares problem Ax ≈ *b,* if the vector *b* lies in span(A), then the residual is 0.
3.6. True or false: In solving a linear least squares problem Ax ≈ b, if the residual is 0, then the solution *x* must be unique.
3.7. True or false: The product of a Householder transformation and a Givens rotation is always an orthogonal matrix.
3.8. True or false: If the n x n matrix Q is a Householder transformation, and x is an arbitrary n-vector, then the last k components of the vector Qx are zero for some $k < n$.
3.9. True or false: Methods based on orthogonal factorization are generally more expensive computationally than methods based on the normal equations for solving linear least squares problems.
3.10. (a) In a data-fitting problem in which *m* data points *(ti, yi)* are fit by a model function *f(t,x),* where t is the independent variable and x is an n-vector of parameters to be determined, what does it mean for the function *f* to be *linear* in the components of x?
(b) Give an example of a model function *f(t,x)* that is linear in this sense.
(c) Give an example of a model function *f(t,x)* that is nonlinear.
3.11. In a linear least squares problem Ax ≈ *b,* where A is an m x n matrix, if rank(A) < n, then which of the following situations are possible?
(a) There is no solution.
(b) There is a unique solution.
(c) There is a solution, but it is not unique.
3.12. In solving an overdetermined least squares problem Ax ≈ *b,* which would be a more serious difficulty: that the rows of A are linearly dependent, or that the columns of A are linearly dependent? Explain.
3.13. In an overdetermined linear least squares problem with model function $f(t,x) = x_1\phi_1(t) + x_2\phi_2(t) + x_3\phi_3(t)$, what will be the rank of the resulting least squares matrix A if we take $\phi_1(t)=1$, $\phi_2(t)= t$, and $\phi_3(t)=1-t$?
3.14. What is the system of normal equations for the linear least squares problem Ax ≈ b?
3.15. List two ways in which use of the normal equations for solving linear least squares problems may suffer loss of numerical accuracy.
3.16. Let A be an m x n matrix. Under what conditions on the matrix A is the matrix A^TA
(a) Symmetric? (b) Nonsingular? (c) Positive definite?
3.17. Which of the following properties of an m x n matrix A, with m > n, indicate that the minimum residual solution of the least squares problem Ax ≈ b is not unique?
(a) The columns of A are linearly dependent. (b) The rows of A are linearly dependent.
(c) The matrix $A^T A$ is singular.
3.18. (a) Can Gaussian elimination with pivoting be used to compute an LU factorization of a rectangular *m x* n matrix A, where *L* is an *m x* k matrix whose entries above its main diagonal

are all zero, U is a k x n matrix whose entries below its main diagonal are all zero, and $k = \min\{m, n\}$?

(b) If this were possible, would it provide a way to solve an overdetermined least squares problem $Ax \approx b$, where $m > n$? Why?

3.19. (a) What is meant by two vectors x and y being orthogonal to each other?

(b) Prove that if two nonzero vectors are orthogonal to each other, then they must also be linearly independent.

(c) Give an example of two nonzero vectors in the plane that are orthogonal to each other.

(d) Give an example of two nonzero vectors in the plane that are not orthogonal to each other.

(e) List two ways in which orthogonality is important in the context of linear least squares problems.

3.20. In Euclidean n-space, is orthogonality a transitive relation? That is, if x is orthogonal to y, and y is orthogonal to z, is x necessarily orthogonal to z?

3.21. What is meant by an orthogonal projector? How is this concept relevant to linear least squares?

3.22. (a) Why are orthogonal transformations, such as Householder or Givens, often used to solve least squares problems?

(b) Why are such methods not often used to solve square linear systems?

(c) Do orthogonal transformations have any advantage over Gaussian elimination for solving square linear systems? If so, state one.

3.23. Which of the following matrices are orthogonal?

(a) [0 1;1 0] (b) [1 0;0 -1] (c) [2 0;0 (1/2)] (d) [sqrt(2)/2 sqrt(2)/2;-sqrt(2)/2 sqrt(2)/2]

3.24. Which of the following properties does an $n \times n$ orthogonal matrix necessarily have?

(a) It is nonsingular.
(b) It preserves the Euclidean vector norm when multiplied times a vector.
(c) Its transpose is its inverse.
(d) Its columns are orthonormal.
(e) It is symmetric.
(f) It is diagonal.
(g) Its Euclidean matrix norm is l.
(h) Its Euclidean condition number l.

3.25. Which of the following types of matrices are necessarily orthogonal?

(a) Permutation
(b) Symmetric positive definite
(c) Householder transformation
(d) Givens rotation
(e) Nonsingular
(f) Diagonal

3.26. Show that multiplication by an orthogonal matrix preserves the Euclidean norm of a vector.

3.27. What condition must a nonzero n-vector w satisfy to ensure that the matrix $H = I - 2ww^T$ is orthogonal?

3.28. If Q is a 2×2 orthogonal matrix such that Q[1 1]'=[α 0]', what must the value of α be?

3.29. How many scalar multiplications are required to multiply an arbitrary n-vector by an $n \times n$ Householder transformation matrix $H = I - 2ww^T$, where w is an n-vector with $\|w\|_2 = 1$?

3.30. Given a vector a, in designing a Householder transformation H such that $Ha = \alpha e_1$, we know that $\alpha = \pm \|a\|_2$. On what basis should the sign be chosen?

3.31. List one advantage and one disadvantage of Givens rotations for QR factorization compared with Householder transformations.
3.32. When used to annihilate the second component of a 2-vector, does a Householder transformation always give the same result as a Givens rotation?
3.33. In addition to the input array containing the matrix A, which can be overwritten, how much additional auxiliary array storage is required to compute and store the following?
(a) The LU factorization of A by Gaussian elimination with partial pivoting, where A is $n \times n$
(b) The QR factorization of A by Householder transformations, where A is $m \times n$
3.34. In solving a linear least squares problem $Ax \approx b$, where A is an $m \times n$ matrix with $m \geq n$ and rank$(A) < n$, at what point will the least squares solution process break down (assuming exact arithmetic)?
(a) Using Cholesky factorization to solve the normal equations
(b) Using QR factorization by Householder transformations
3.35. Compared to the classical Gram-Schmidt procedure, which of the following are advantages of modified Gram-Schmidt orthogonalization?
(a) Requires less storage (b) Requires less work (c) Is more stable numerically.
3.36. For computing the QR factorization of an $m \times n$ matrix, with $m \geq n$, how large must n be before there is a difference between the classical and modified Gram-Schmidt procedures?
3.37. Explain why the Householder method requires less storage than the modified Gram-Schmidt method for computing the QR factorization of a matrix A.
3.38. Explain how QR factorization with column pivoting can be used to determine the rank of a matrix.
3.39. Explain why column pivoting can be used with the modified Gram-Schmidt orthogonalization procedure but not with the classical Gram-Schmidt procedure.
3.40. In terms of the condition number of the matrix A compare the range of applicability of the normal equations method and the Householder QR method for solving the linear least squares problem $Ax \approx b$ [i.e., for what values of cond(A) can each method be expected to break down?].
3.41. Let A be an $m \times n$ matrix.
(a) What is the maximum number of nonzero singular values that A can have?
(b) If rank$(A) = k$, how many nonzero singular values does A have?
3.42. Let a be a nonzero column vector. Considered as an $n \times 1$ matrix, a has only one positive singular value. What is its value?
3.43. Express the Euclidean condition number of a matrix in terms of its singular values.
3.44. List two reliable methods for determining the rank of a rectangular matrix numerically.
3.45. If A is a $2n \times n$ matrix, rank the following methods according to the amount of work required to solve the linear least squares problem $Ax \approx b$.
(a) QR factorization by Householder transformations (b) Normal equations (c) Singular value decomposition.
3.46. List at least two applications for the singular value decomposition (SVD) of a matrix other then solving least squares problems.