REVIEW QUESTIONS - CHAPTER 2

2.1. True or false: If a matrix A is nonsingular, then the number of solutions to the linear system $Ax = b$ depends on the particular choice of right-hand-side vector b.
2.2. True or false: If a matrix has a very small determinant, then the matrix is nearly singular.
2.3. True or false: For a symmetric matrix A, it is always the case that $|| A ||_1 = || A ||_\infty$
2.4. True or false: If a triangular matrix has a zero entry on its main diagonal, then the matrix is necessarily singular.
2.5. True or false: If a matrix has a zero entry on its main diagonal, then the matrix is necessarily singular.
2.6. True or false: An underdetermined system of linear equations $Ax = b,$ where A is an $m \times n$ matrix with $m < n,$ always has a solution.
2.7. True or false: The product of two upper triangular matrices is upper triangular.
2.8. True or false: The product of two symmetric matrices is symmetric.
2.9. True or false: The inverse of a nonsingular upper triangular matrix is upper triangular.
2.10. True or false: If the rows of an $n \times n$ matrix A are linearly dependent, then the columns of the matrix are also linearly dependent.
2.11. True or false: A system of linear equations $Ax = b$ has a solution if, and only if, the $m \times n$ matrix A and the augmented $m \times (n + 1)$ matrix $[A\ b]$ have the same rank.
2.12. True or false: If A is any $n \times n$ matrix and P is any $n \times n$ permutation matrix, then $PA = AP$.
2.13. True or false: Provided row interchanges are allowed, the LU factorization always exists, even for a singular matrix A.
2.14. True or false: If a linear system is well-conditioned, then pivoting is unnecessary in Gaussian elimination.
2.15. True or false: If a matrix is singular then it cannot have an LU factorization.
2.16. True or false: If a nonsingular symmetric matrix is not positive definite, then it cannot have a Cholesky factorization.
2.17. True or false: A symmetric positive definite matrix is always well-conditioned.
2.18. True or false: Gaussian elimination without pivoting fails only when the matrix is ill-conditioned or singular.
2.19. True or false: Once the LU factorization of a matrix has been computed to solve a linear system, then subsequent linear systems with the same matrix but different right-hand-side vectors can be solved without refactoring the matrix.
2.20. True or false: In explicit matrix inversion by LU factorization and triangular solution, the majority of the work is due to the factorization.
2.21. True or false: If x is any n-vector, then $\| x \|_1 \geq \| x \|_\infty$
2.22. True or false: The norm of a singular matrix is zero.
2.23. True or false: If $\| A \| = 0$, then $A = 0$.
2.24. True or false: $\| A \|_1 = \| A^T \|_\infty$.
2.25. True or false: If A is any $n \times n$ nonsingular matrix, then $\text{cond}(A) = \text{cond}(A^{-1})$.
2.26. True or false: In solving a nonsingular system of linear equations, Gaussian elimination with partial pivoting usually yields a small residual even if the matrix is ill-conditioned.
2.27. True or false: The multipliers in Gaussian elimination with partial pivoting are bounded by 1 in magnitude, so the entries of the successive reduced matrices cannot grow in magnitude.
2.28. Can a system of linear equations $Ax = b$ have exactly two distinct solutions?

2.29. Can the number of solutions to a linear system $Ax = b$ ever be determined solely from the matrix A without knowing the right-hand-side vector b?
2.30. In solving a square system of linear equations $Ax = b$, which would be a more serious difficulty: that the rows of A are linearly dependent, or that the columns of A are linearly dependent? Explain.
2.31. (a) State one defining property of a *singular* matrix A.

(b) Suppose that the linear system $Ax = b$ has two distinct solutions x and y. Use the property you gave in part *a* to prove that A must be singular.

2.32. Given a nonsingular system of linear equations $Ax = b$, what effect on the solution vector x results from each of the following actions?

(a) Permuting the rows of $[A \;\; b]$

(b) Permuting the columns of A

(c) Multiplying both sides of the equation from the left by a nonsingular matrix M

2.33. Suppose that both sides of a system of linear equations $Ax = b$ are multiplied by a nonzero scalar α.

(a) Does this change the true solution x?

(b) Does this change the residual vector $r = b - Ax$ for a given x?

(c) What conclusion can be drawn about assessing the quality of a computed solution?

2.34. Suppose that both sides of a system of linear equations $Ax = b$ are premultiplied by a nonsingular diagonal matrix.

(a) Does this change the true solution x?

(b) Can this affect the conditioning of the system?

(c) Can this affect the choice of pivots in Gaussian elimination?

2.35. Specify an elementary elimination matrix that zeros the last two components of the vector: [3 2 -1 4]'.
2.36. (a) Specify a 4 x 4 permutation matrix that interchanges the 2nd and 4th components of any 4-vector.

(c) Specify a 4 x 4 permutation matrix that reverses the order of the components of any 4-vector.

2.37. With a singular matrix and the use of exact arithmetic, at what point will the solution process break down in solving a linear system by Gaussian elimination?

(a) With partial pivoting?

(b) Without pivoting?

2.38. (a) What is the difference between partial pivoting and complete pivoting in Gaussian elimination?

(b) State one advantage of each type of pivoting relative to the other.

2.39. Consider the following matrix A, whose LU factorization we wish to compute using Gaussian elimination: A = [4 -8 1;6 5 7;0 -10 -3].
What will the initial pivot element be if

(a) No pivoting is used?

(b) Partial pivoting is used?

(c) Complete pivoting is used?

2.40. Give two reasons why pivoting is essential for a numerically stable implementation of Gaussian elimination.
2.41. If A is an ill-conditioned matrix, and its LU factorization is computed by Gaussian elimination with partial pivoting, would you expect the ill-conditioning to be reflected in L, in U, or both? Why?

2.42. (a) What is the inverse of the following matrix? [1 0 0 0;0 1 0 0;0 m1 1 0;0 m2 0 1]
(b) How might such a matrix arise in computational practice?
2.43. (a) Can every nonsingular $n \times n$ matrix A be written as a product, $A = LU$, where L is a lower triangular matrix and U is an upper triangular matrix?

(b) If so, what is an algorithm for accomplishing this? If not, give a counterexample to illustrate.
2.44. Given an $n \times n$ nonsingular matrix A and a second $n \times n$ matrix B, what is the best way to compute the $n \times n$ matrix $A^{-1}B$?
2.45. If A and B are $n \times n$ matrices, with A nonsingular, and c is an n-vector, how would you efficiently compute the product $A^{-1}Bc$?
2.46. If A is an $n \times n$ matrix and x is an n-vector, which of the following computations requires less work? Explain.
(a) $y = (x\,x^T)\,A$
(b) $y = x\,(x^T A)$
2.47. How does the computational work in solving an $n \times n$ triangular system of linear equations compare with that for solving a general $n \times n$ system of linear equations?
2.48. Assume that you have already computed the LU factorization, $A = LU$, of the nonsingular matrix A. How would you use it to solve the linear system $A^T x = b$?
2.49. If L is a nonsingular lower triangular matrix, P is a permutation matrix, and b is a given vector, how would you solve each of the following linear systems?
(a) $LPx = b$
(b) $PLx = b$
2.50. In the plane $\mathbf{R}^2$, is it possible to have a vector $x \neq 0$ such that $\| x \|_1 = \| x \|_\infty$? If so, give an example.
2.51. In the plane $\mathbf{R}^2$, is it possible to have two vectors x and y such that $\| x \|_1 > \| y \|_1$ but $\| x \|_\infty < \| y \|_\infty$? If so, give an example.
2.52. In general, which matrix norm is easier to compute, $\| A \|_1$ or $\| A \|_2$? Why?
2.53. (a) Is the magnitude of the determinant of a matrix a good indicator of whether the matrix is nearly singular .
(b) If so, why? If not, what is a better indicator of near singularity?
2.54. (a) How is the condition number of a matrix A defined for a given matrix norm?
(b) How is the condition number used in estimating the accuracy of a computed solution to a linear system $Ax = b$?
2.55. Why is computing the condition number of a general matrix a nontrivial problem?
2.56. Give an example of a 3×3 matrix A, other than the identity matrix, such that cond(A) = 1.
2.57. (a) What is the condition number of the following matrix using the 1-norm?
[4 0 0;0 -6 0;0 0 2] (b) Does your answer differ using the ∞-norm?
2.58. Suppose that the $n \times n$ matrix A is perfectly well-conditioned, i.e., cond(A) = 1. Which of the following matrices would then necessarily share this same property?
(a) cA, where c is any nonzero scalar
(b) DA, where D is any nonsingular diagonal matrix
(c) PA, where P is any permutation matrix
(d) BA, where B is any nonsingular matrix
(e) A^{-1}, the inverse of A
(f) A^T, the transpose of A
2.59. Let A = diag(½) be an $n \times n$ diagonal matrix with all its diagonal entries equal to ½.

(a) What is the value of det(A)?

(b) What is the value of cond(A)?

(c) What conclusion can you draw from these results?

2.60. Suppose that the $n \times n$ matrix A is exactly singular, but that its floating-point representation, fl(A), is nonsingular. In this case, what would you expect the order of magnitude of the condition number cond(fl(A)) to be?

2.61. Classify each of the following matrices as well-conditioned or ill-conditioned:

(a) $[10^{10}\ 0;0\ 10^{-10}]$ (b) $[10^{10}\ 0;0\ 10^{10}]$ (c) $[10^{-10}\ 0;0\ 10^{-10}]$ (d) [1 2;2 4]

2.62. Which of the following are good indicators that a matrix is nearly singular?

(a) Its determinant is small.

(b) Its norm is small.

(c) Its norm is large.

(d) Its condition number is large.

2.63. (a) In solving a linear system Ax = b, what is meant by the residual of an approximate solution x?

(b) Does a small relative residual always imply that the solution is accurate? Why?

(c) Does a large relative residual always imply that the solution is inaccurate? Why?

2.64. In a floating-point system having 10 decimal digits of precision, if Gaussian elimination with partial pivoting is used to solve a linear system whose matrix has a condition number of 10^3, and whose input data are accurate to full machine precision, about how many digits of accuracy would you expect in the solution?

2.65. Assume that you are solving a system of linear equations $Ax = b$ on a computer whose floating-point number system has 12 decimal digits of precision, and that the problem data are correct to full machine precision. About how large can the condition number of the matrix A be before the computed solution x will contain no significant digits?

2.66. Under what circumstances does a small residual vector $r = b - Ax$ imply that x is an accurate solution to the linear system $Ax = b$?

2.67. Let A be an arbitrary square matrix and c an arbitrary scalar. Which of the following statements must necessarily hold?

(a) $\| cA \| = | c | \cdot \| A \|$ (b) cond(cA) = $| c | \cdot$ cond(A).

2.68. (a) What is the main difference between Gaussian elimination and Gauss-Jordan elimination?

(b) State one advantage of each type of elimination relative to the other.

2.69. Rank the following methods according to the amount of work required for solving a general system of linear equations of order n:

(a) Gauss-Jordan elimination

(c) Gaussian elimination with partial pivoting

(d) Cramer's rule

(d) Explicit matrix inversion followed by matrix-vector multiplication

2.70. (a) How much storage is required to store an $n \times n$ matrix of rank one efficiently?

(b) How many arithmetic operations are required to multiply an n-vector by an $n \times n$ matrix of rank one efficiently?

2.71. In a comparison of ordinary Gaussian elimination with Gauss-Jordan elimination for solving a linear system $Ax = b$,

(a) Which has a more expensive factorization?

(b) Which has a more expensive back-substitution?

(c) Which has a higher cost overall?

2.72. For each of the following elimination algorithms for solving linear systems, is there any pivoting strategy that can guarantee that all of the multipliers will be at most 1 in absolute value?

(a) Gaussian elimination (b) Gauss-Jordan elimination

2.73. What two properties of a matrix A together imply that A has a Cholesky factorization?

2.74. List three advantages of Cholesky factorization compared with LU factorization.

2.75. How many square roots are required to compute the Cholesky factorization of an $n \times n$ symmetric positive definite matrix?

2.76. Let $A = \{a_{ij}\}$ be an $n \times n$ symmetric positive definite matrix.

(a) What is the (1, 1) entry of its Cholesky factor?

(b) What is the (n, 1) entry of its Cholesky factor?

2.77. What is the Cholesky factorization of the following matrix? [4 2;2 2]

2.78. (a) Is it possible, in general, to solve a symmetric indefinite linear system at a cost similar to that for using Cholesky factorization to solve a symmetric positive definite linear system?

(b) If so, what is an algorithm for accomplishing this? If not, why?

2.79. Give two reasons why iterative improvement for solutions of linear systems is often impractical to implement.

2.80. Suppose you have already solved the $n \times n$ linear system $Ax = b$ by LU factorization and back-substitution. What is the further cost (order of magnitude will suffice) of solving a new system

(a) With the same matrix A but a different right-hand-side vector?

(b) With the matrix changed by adding a matrix of rank one?

(c) With the matrix A changed completely?