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Homework number: 7
Homework Title: Exercise 8.4

Problem description:
Fill in the details of the derivation of the error estimates for the midpoint and trapezoid quadrature rules given in Section 8.3.1. In particular, show that the odd-order terms drop out in both cases, as claimed.

Problem solution:
a) midpoint quadrature rule

$$f(x) = f'(m)(x-m) + \frac{f''(m)}{2}(x-m)^2 + \frac{f^{(3)}(m)}{6}(x-m)^3 + \frac{f^{(4)}(m)}{24}(x-m)^4$$

Integrating this expression we get:

$$\int_a^b f(x)dx = f'(m)\int_a^b (x-m)dx + \frac{f''(m)}{2}\int_a^b (x-m)^2 dx +$$

$$+\frac{f^{(3)}(m)}{6}\int_a^b (x-m)^3 dx + \frac{f^{(4)}(m)}{24}\int_a^b (x-m)^4 dx$$

First interesting term:

$$\int_a^b (x-m)dx = (\frac{x^2}{2} - xm)|_a^b = (\frac{x^2}{2} - x\frac{a+b}{2})|_a^b =$$

$$= \frac{b^2 - ba - b^2 - a^2 + a^2 + ab}{2} = 0$$

Second interesting term:

$$\int_a^b (x-m)^3 dx = \frac{(x-m)^4}{4}|_a^b = \frac{(x-\frac{a+b}{2})^4}{4}|_a^b =$$

$$\frac{(b-\frac{a+b}{2})^4}{4} - \frac{(a-\frac{a+b}{2})^4}{4} =$$

When we now split up the polynomials $(a^2 - b^2 = (a-b)(a+b))$

$$\frac{((b-\frac{a+b}{2})^2 - (a-\frac{a+b}{2})^2)\dots}{4} = \frac{(\frac{b-a}{2} - \frac{a-b}{2})(\frac{b-a}{2} + \frac{a-b}{2})\dots}{4} =$$

$$\frac{(\frac{b-a-a+b}{2})(\frac{b-a+a-b}{2})\dots}{4} = 0$$

Higher odd-order terms can be solved the same way and drop out.

b) trapezoid quadrature rule
We got the Taylor series as before

$$f(x) = f'(m)(x-m) + \frac{f''(m)}{2}(x-m)^2 + \frac{f^{(3)}(m)}{6}(x-m)^3 + \frac{f^{(4)}(m)}{24}(x-m)^4$$

Now we substitute for x=a and x=b

$$I : f(a) = f'(m)(a-m) + \frac{f''(m)}{2}(a-m)^2 + \frac{f^{(3)}(m)}{6}(a-m)^3 + \frac{f^{(4)}(m)}{24}(a-m)^4$$

$$II : f(b) = f'(m)(b-m) + \frac{f''(m)}{2}(b-m)^2 + \frac{f^{(3)}(m)}{6}(b-m)^3 + \frac{f^{(4)}(m)}{24}(b-m)^4$$

Adding these two expressions together we get

$$f(a) + f(b) = 2f(m) + f'(m)((a-m) + (b-m)) + \frac{f''(m)}{2}((a-m)^2 + (b-m)^2) +$$

$$+\frac{f^{(3)}(m)}{6}((a-m)^3 + (b-m)^3) + \frac{f^4(m)}{24}((a-m)^4 + (b-m)^4)$$

Therefore we got the first interesting term:

$$(a-m) + (b-m) = a + b - 2m = a + b - 2(a+b)/2 = 0$$

Second interesting term:

$$(a-m)^3 + (b-m)^3 = a^3 + b^3 - 3m(a^2 + b^2) + 3m^2(a+b) - 2m^3 =$$

$$a^3 + b^3 - 3(\frac{a+b}{2})(a^2 + b^2) + 3(\frac{a+b}{2})^2(a+b) - 2(\frac{a+b}{2})^3 =$$

$$= -\frac{(a+b)^3}{2} + 3\frac{(a+b)^3}{4} - 2\frac{(a+b)^3}{8} = \frac{(a+b)^3(-2+3-1)}{4} = 0$$

Higher odd-order terms can be solved the same way and drop out too.

Results:

The odd order terms drop out as claimed in both midpoint and trapezoid quadrature rule.