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Homework number: 3
Homework Title: Exercise 3.1

Problem description:

If a vertical beam has a downward force applied at its lower end, the amount by which it stretches will be proportional to the magnitude of the force. Thus, the total length y of the beam is given by the equation

$$y = x_1 + x_2 * t,$$

where x_1 is its original length, t is the force applied, and x_2 is the proportionality constant. Suppose that the following measurements are taken:

t	10	15	20
y	11.60	11.85	12.25

- (a) Set up the over determined 3×2 system of linear equations corresponding to the data collected.
- (b) Is this system consistent? If not, compute each possible pair of values for (x_1, x_2) obtained by selecting any two of the equations from the system. Is there any reason to prefer any one of these results?
- (c) Set up the system of normal equations and solve it to obtain the least squares solution to the over determined system. Compare your result with those obtained in part (b).

Problem solution:

- (a) Set up system as described in notes
- (b) Select two equations, eliminate one unknown, calculate the remaining and get the other one by back substitution. Do this for all 3 possibilities.
- (c) System of normal equations $\mathbf{A}^T * \mathbf{A} * \mathbf{x} = \mathbf{A}^T * \mathbf{b}$ has the same solution as linear least squares problem $\mathbf{A}^T * \mathbf{x} \simeq \mathbf{b}$. From this system we get $\mathbf{x} = (\mathbf{A}^T * \mathbf{A})^{-1} * \mathbf{A}^T * \mathbf{b}$, which can be calculated easily.

Results:

$$(a) A * x = \begin{bmatrix} 1 & 10 \\ 1 & 15 \\ 1 & 20 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \simeq \begin{bmatrix} 11.6 \\ 11.85 \\ 12.25 \end{bmatrix} = b$$

(b) 3 equations:

1. $x_1 + 10 * x_2 = 11.6$
2. $x_1 + 15 * x_2 = 11.85$
3. $x_1 + 20 * x_2 = 12.25$

3 possible combinations:

- 1) - 2): $-5 * x_2 = -0.25 \Rightarrow x_2 = 0.05, x_1 = 11.1$
- 1) - 3): $-10 * x_2 = -0.65 \Rightarrow x_2 = 0.065, x_1 = 10.95$
- 2) - 3): $-5 * x_2 = -0.4 \Rightarrow x_2 = 0.08, x_1 = 10.65$

We can prefer the 2nd solution, because for the overall solution this one might be the best approximation: x_2 is exactly and x_1 is nearly the average when combining all 3 x_2, x_1 .

$$(c) \begin{aligned} &\bullet A^T * A = \begin{bmatrix} 1 & 1 & 1 \\ 10 & 15 & 20 \end{bmatrix} * \begin{bmatrix} 1 & 10 \\ 1 & 15 \\ 1 & 20 \end{bmatrix} = \begin{bmatrix} 3 & 45 \\ 45 & 725 \end{bmatrix} \\ &\bullet (A^T * A)^{-1} = \begin{bmatrix} 4.8333 & -0.3 \\ -0.3 & 0.02 \end{bmatrix} \\ &\bullet A^T * b = \begin{bmatrix} 1 & 1 & 1 \\ 10 & 15 & 20 \end{bmatrix} * \begin{bmatrix} 11.6 \\ 11.85 \\ 12.25 \end{bmatrix} = \begin{bmatrix} 35.7 \\ 538.75 \end{bmatrix} \\ &\bullet \Rightarrow x = \begin{bmatrix} 4.8333 & -0.3 \\ -0.3 & 0.02 \end{bmatrix} * \begin{bmatrix} 35.7 \\ 538.75 \end{bmatrix} = \begin{bmatrix} 10.925 \\ 0.0650 \end{bmatrix} \end{aligned}$$

The result is nearly the same as the one of the second solution in (b).

Discussion and Comments:

The second solution in (b) is the best approximation of the overall approximated solution.